3.8 Problem set 3

Problem 3.1: Unitary operators and states normalisation

Let $\hat{U}$ be a unitary operator, and let $|\psi\rangle$ be the state of a quantum system. Show that this unitary preserves normalisation, i.e.,

$$\| \, |\psi\rangle \, \| = \| \, \hat{U} \, |\psi\rangle \, \| = 1 \; .$$

Problem 3.2: Unitarity of Pauli operators

Show that Pauli operators, $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ are unitary.

Problem 3.3: Eigenvalues and eigenvectors of Pauli operators

Find the eigenvalues and normalised eigenvectors of Pauli operators, $\hat{X}$, $\hat{Y}$, and $\hat{Z}$.

Problem 3.4: Square of Pauli matrices

1. Show that, for $\hat{A} \in \{\hat{X}, \hat{Y}, \hat{Z}\}$,
$$\hat{A}^2 = \mathbb{1} \; .$$
2. Use the previous result and show that
$$\mathrm{e}^{i\theta \hat{A}} = \cos\theta \; \mathbb{1} \; + \; i \sin\theta \; \hat{A} \; .$$
3. We define the vector of Pauli matrices $\vec{\hat{\sigma}}$ and the unit vector $\vec{n}$ by
$$\vec{\hat{\sigma}} = \begin{pmatrix} \hat{X} \\ \hat{Y} \\ \hat{Z} \end{pmatrix} \; ,$$
$$\vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \; .$$
   (a) Prove that
   $$\left(\vec{n} \cdot \vec{\hat{\sigma}}\right)^2 = \mathbb{1} \; .$$
   (b) Deduce the expression of the operator $\hat{R}_\theta$, defined by
   $$\hat{R}_\theta = \mathrm{e}^{\frac{-i\theta}{2} \; \vec{n} \cdot \vec{\hat{\sigma}}} \; .$$

Problem 3.5: Eigenstates of Hadamard gate

Show that the eigenstates of the Hadamard gate are given in the computational basis as

$$|v_{\pm}\rangle = \cos\frac{\pi}{8}\ |0\rangle\ \pm\ \sin\frac{\pi}{8}\ |1\rangle\ .$$

Problem 3.6: Circuit identities

Prove the following identities:

1. $$\hat{H}\hat{X}\hat{H} = \hat{Z}\ .$$
2. $$\hat{H}\hat{Y}\hat{H} = -\hat{Y}\ .$$
3. $$\hat{H}\hat{Z}\hat{H} = \hat{X}\ .$$

Problem 3.7: Controlled gates

Find the matrix representations of the following controlled gates:

1. controlled-$\hat{Y}$.
2. controlled-$\hat{Z}$.
3. controlled-$\hat{H}$.

Problem 3.8: Quantum teleportation

We consider the quantum circuit shown in **Figure** 3.6. Show that $|\psi'\rangle = |\psi\rangle$. In the aforementioned circuit, the state φ is the EPR state given by $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

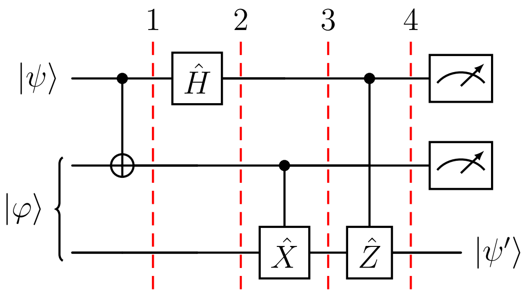

Figure 3.6: (Problem 3.6) Quantum teleportation circuit.