1.6 Problem set 1

Problem 1.1: Matrix representation of an operator

Let $\mathcal{H}$ be the Hilbert space associated to a physical system. Let $\{|1\rangle, |2\rangle, |3\rangle\}$ be an orthonormal basis for $\mathcal{H}$.

$\hat{A}$ is an operator in $\mathcal{H}$ such that

$$\hat{A}|1\rangle = |2\rangle \ , \tag{1.59a}$$

$$\hat{A}|2\rangle = \frac{1}{\sqrt{5}}(|1\rangle + 2i|3\rangle) \, , \tag{1.59b}$$

$$\hat{A}|3\rangle = |1\rangle \ . \tag{1.59c}$$

Find the matrix representation of the operator $\hat{A}$ in the basis $\{|1\rangle, |2\rangle, |3\rangle\}$.

Problem 1.2: Matrix of the product of operators

Let $\mathcal{H}$ be the Hilbert space for some system, and let $\{|1\rangle, |2\rangle\}$ be an orthonormal basis for $\mathcal{H}$.

$\hat{A}$ and $\hat{B}$ are two operators acting on vectors of $\mathcal{H}$. Suppose that $\hat{A}$ and $\hat{B}$ act on the basis vectors as

$$\hat{A}|1\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \ , \tag{1.60a}$$

$$\hat{A}|2\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \ , \tag{1.60b}$$

$$\hat{B}|1\rangle = |1\rangle \ , \tag{1.60c}$$

$$\hat{B}|2\rangle = \frac{1}{\sqrt{2}}(|1\rangle + (1+i)|2\rangle) \ . \tag{1.60d}$$

1. Find the matrix representation of the operators $\hat{A}$ and $\hat{B}$.
2. Show that the matrix representation of the product $\hat{A}\hat{B}$ is equal to the product of the two matrices representing $\hat{A}$ and $\hat{B}$.
3. Find the matrix of the operator $\hat{B}\hat{A}$.
4. Deduce the matrix of the commutator $[\hat{A}, \hat{B}]$.

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Problem 1.3: Matrix representation of the identity operator

Consider the orthonormal basis $\{|1\rangle, \dots, |d\rangle\}$ of the Hilbert space $\mathcal{H}$ for a qudit of dimension d.

Show that the matrix of the identity operator $\mathbb{1}$ acting on the space $\mathcal{H}$ is the identity matrix, i.e., the matrix that has all elements equal to zero, except the diagonal elements which are all equal to 1.

Problem 1.4: Orthogonal vectors

Consider a Hilbert space with the orthonormal basis $\{|1\rangle, |2\rangle\}$. Suppose that we have the vectors,

$$|\psi\rangle = \alpha\,|1\rangle + \beta\,|2\rangle \ , \tag{1.61a}$$

$$|\varphi\rangle = \gamma\,(|1\rangle + i\,|2\rangle) \ , \tag{1.61b}$$

where α, β, and γ are complex parameters.

1. What should α, β, and γ satisfy, such that the vectors $|\psi\rangle$ and $|\varphi\rangle$ are normalised (i.e., are unit vectors)?
2. What are all the parametrised vectors $|\psi\rangle$ that are
   (a) orthogonal to $|\varphi\rangle$?
   (b) linearly dependent with $|\varphi\rangle$?

Problem 1.5: Hermitian conjugate of an operator

Let $|\psi\rangle$ and $|\varphi\rangle$ be two arbitrary vectors in a Hilbert space $\mathcal{H}$.

1. Show that $|\psi\rangle\langle\varphi|$ is a linear operator in $\mathcal{H}$.
2. Show that:

$$(|\psi\rangle\langle\varphi|)^\dagger = |\varphi\rangle\langle\psi| \ . \tag{1.62}$$

Problem 1.6: The projector operator

We consider the state vector $|\psi\rangle$ from the Hilbert space $\mathcal{H}$. The projector operator on the state $|\psi\rangle$ is the operator

$$\hat{P}_\psi = |\psi\rangle\langle\psi| \ . \tag{1.63}$$

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1. Show that $\hat{P}_\psi$ is a Hermitian operator.
2. Show that $\hat{P}_\psi^2 = \hat{P}_\psi$.
3. Find the operator $\mathrm{e}^{i\theta\hat{P}_\psi}$.

Problem 1.7: Eigenvalues of a Hermitian operator

Show that all eigenvalues of a Hermitian operator are real.

Problem 1.8: Orthogonality of eigenvectors

Show that eigenvectors of a Hermitian operator, with distinct eigenvalues, are necessarily orthogonal.

Problem 1.9: Evolution Operator

The Schrödinger equation is written as

$$i\hbar\frac{\mathrm{d}\left|\psi(t)\right\rangle}{\mathrm{d}t} = \hat{H}(t)\left|\psi(t)\right\rangle \ ,$$

where $|\psi(t)\rangle$ is the state of system at time t, and $\hat{H}$ is the Hamiltonian operator.

Show that, if the Hamiltonian is time-independent, the time evolution of the system's state is given by

$$\left|\psi(t)\right\rangle = \hat{U}(t_0, t)\left|\psi(t_0)\right\rangle \ , \tag{1.64}$$

where $|\psi(t_0)\rangle$ is the state of the system at the initial time t_0, and $\hat{U}$ is a unitary operator given by the expression,

$$\hat{U}(t_0, t) = \mathrm{e}^{\frac{-i(t-t_0)}{\hbar}\ \hat{H}} \ . \tag{1.65}$$

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